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# Modelling the Social Dynamics of a Sex Industry: Its Implications for Spread of HIV/AIDS

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A theoretical model is proposed for a community which has the structure of two classes (direct and indirect) of commercial sex workers (CSW), and two classes of sexually active male customers with different levels of sexual activity. The direct CSW's work in brothels while the indirect CSW's are based in commercial establishments such as bars, cafes, and massage parlours where sex can be bought on request and conducted elsewhere. Behaviour change and the resulting change of activity class occurs between the two classes of CSW's and two classes of males under the setting of the proliferation of human immunodeficiency virus (HIV)/acquired immunodeficiency syndrome epidemic and the subsequent intervention programmes. In recently years, this phenomenon has been observed in several countries in Asia. Given the lower levels of condom use and higher HIV prevalence of the indirect CSW's, ascertaining the impact of this change in the structure of the sex industry on the spread of HIV is the main focus of this paper. The complete analysis of the disease-free model is given. For the full model, local analysis will be performed for the case of preferred mixing without activity class change, as well as the case with activity class change and restricted mixing. The basic reproduction number for the spread of epidemic will be derived for each case. Results of biological significance include: (i) the change of behaviour by the CSW's has a more direct effect on the spread of HIV than that of the male customers; (ii) the basic reproduction number is obtained by considering all possible *infection cycles* of the heterosexual transmission of HIV which indicates the importance of understanding the sexual networking in heterosexual transmission of HIV; (iii) the social dynamics of the sex industry is not just a simple 'supply and demand' mechanism driven by the demand of the customers, hence highlighting the need for further understanding of the changing structure of the sex industry. The main points of this work will be illustrated with numerical examples using the HIV data of Thailand.



## 1. INTRODUCTION

Among many Asian countries, the spread of human immunodeficiency virus/acquired immunodeficiency syndrome (HIV/AIDS) has been closely linked with the

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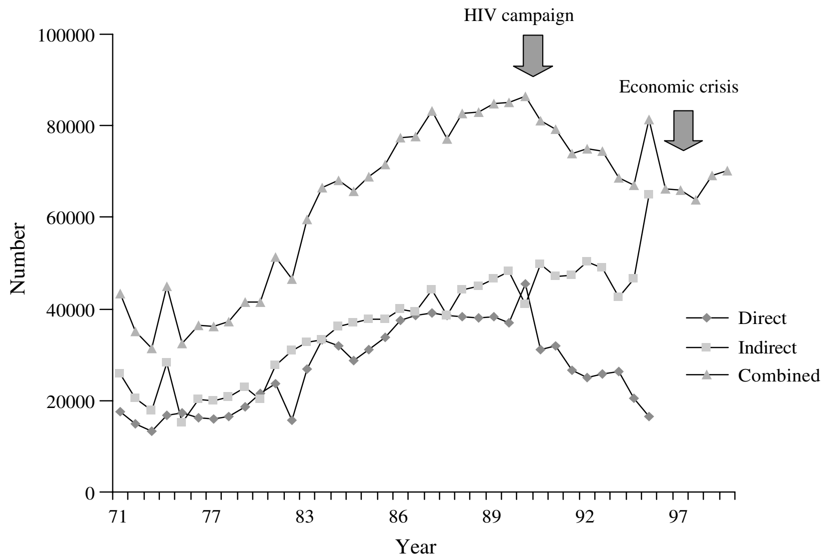
structure of the sex industry (Brown and Xenos, 1994). The case of Thailand has been most explosive and most well-documented. The key ingredients to the epidemic in Thailand and its neighbouring countries (Cambodia, Vietnam, etc.) are a 'core group' of highly sexually active individuals [the female commercial sex workers, or CSW's, see e.g., Weniger *et al*. (1991)] which spreads the disease and a 'bridge population' of unpartnered young men, mainly low-income male brothel visitors, who provide a bridge between the core group and the general population (Bhassorn *et al*., 1993; Morris *et al*., 1996). A model was proposed by Busenberg *et al*. (1995) aimed at studying the importance of the CSW's, although in that work the bridge population was expanded to include all unpartnered young men for the purpose of simplifying the model. The results showed that, among others, the recruitment rate of the CSW's and the relative difference in turnover rate (by death and retirement) of the CSW's once they become infected are important factors in determining whether the disease will persist. More recently, Hsieh and Cooke (2000) propose a model to further study the possible effect of treatment and behaviour change in the society with the above-mentioned situation. [For a review of mathematical modelling of infectious diseases, see Anderson and May (1992).]

In both of these studies, there are only one group of CSW's and one group of male customers. However, in most of these Asian countries the structure of the sex industry is more complicated. Let us consider the case of Thailand, since it is most well-documented [e.g., Wathinee and Guest (1994)]. There are two main forms of commercial sex in these countries, one is brothel-based (direct) and the other nonbrothel-based (indirect) sex establishments which includes bars, cafes, massage parlours, nightclubs etc., where business can be negotiated but not conducted on site [see, e.g., Bhassorn *et al*. (1993), Wathinee and Guest (1994)]. From 1971 on, the Division of Venereal Diseases of the Thai Ministry of Public Health (MOPH) kept separate annual counts of the direct and indirect CSW's (see Fig. 1). However, since 1996 there are no separate counts of the direct and indirect CSW's given due to the lack of reliable data (Pachara Sirivongrangson, personal communication). Recent data in Thailand strongly suggests that many brothel-based CSW's have changed their workplace to bars, cafes, etc. (Thai Working Group on HIV/AIDS Projection, 2001), most likely due to the decrease in brothel business which was brought on by the consistently high HIV prevalence rate among brothel CSW's in the last decade and the economic crisis in 1997–1998. The net result is more indirect sex which is more diverse in patterns and more covert in nature (Hsieh, 2002).

Moreover, the difference in the two forms of commercial sex also leads to discrepancy in the customers who would visit a brothel or an indirect sex establishment. The brothels are more frequented by truck drivers, factory workers, and other low-income men who are more highly active and more likely to visit the brothels. On the other hand, the less frequent customers are more likely to visit the indirect sex establishments. In a study by Napaporn *et al*. (1992), 34% of Thai blue collar men surveyed had 20 or more commercial sex episodes in the previous year and



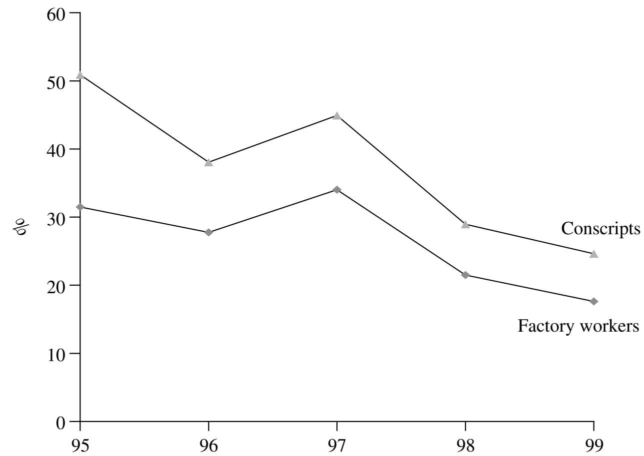

Source: Division of Venereal Diseases, Thai MOPH.
Remark: The census count were carried out semi-annually in 1983–1994

Figure 1. Number of female sex workers in Thailand, 1971–2000.

only 25% had less than 5, while for the white collar men surveyed the figures are 19% with 20 or more and 36% with less than 5. For the same group of men, 40% of blue collar males visited a brothel for their last commercial sex compared to only 19% of the white collar males. Hence the size of different types of customers is also important in the dynamics of the social interaction of the sex industry.

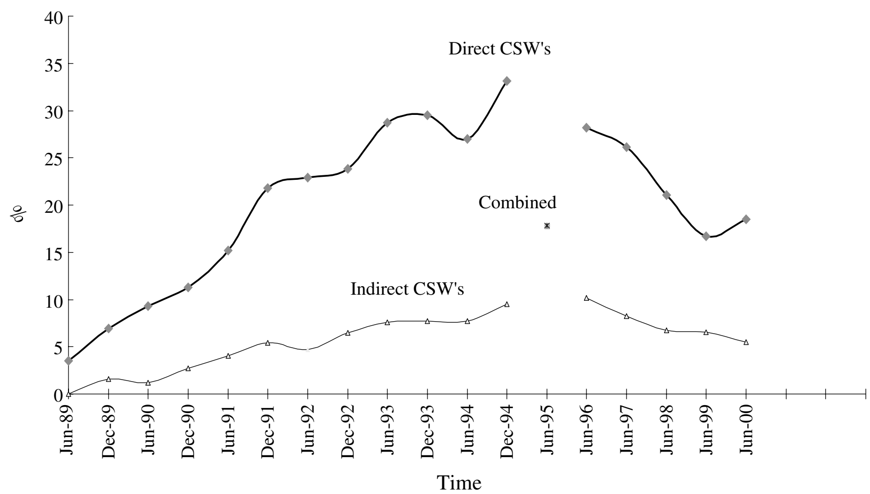
In recent years, the Thai MOPH behavioural sentinel data shows a decrease in the number of customers of the sex industry (Fig. 2). However, the available data (e.g., Fig. 1) seems to indicate that, facing the decrease in demand for brothel workers, many of the direct CSW's have changed to work in nonbrothel establishments in search of better and more profitable business. The mixed message points to a need for further study aimed specifically at understanding this recent change which could impact on the future spread of HIV/AIDS in these countries.

In this work, we propose a more complete model to study specifically the behaviour changes of the two types of CSW's in reaction to the demands for sexual contacts by the groups of male customers. We do so by separating the CSW's into direct and indirect groups, and the male customers into two groups by their levels of frequency to use commercial sex. The significant difference between the two groups in terms of HIV prevalence, frequency of sexual contacts, social make-up of customers, and use of condoms is well-documented [see, e.g., Bhassorn *et al.* (1993) or van Griensven *et al.* (1995)]. Government seroprevalence sentinel data in the past decade have consistently showed the prevalence rate among direct CSW's to be three to four times higher than that of indirect CSW's (see Fig. 3).




Source: Behaviour sentinel surveillance in 19 provinces, Div of Epidemiology, Thai MOPH.

Figure 2. Percentage of customers frequenting sex workers among sentinel male population in Thailand, 1995–1999.




Source: Division of Epidemiology, Thai MOPH.
Remark: Switching from bi-annually (June and December) to annually in June since 1995. June 1996 combined total only.

Figure 3. HIV prevalence among CSWs in Thailand 1989–2000.

Moreover, the most recent HIV sentinel data shows a significantly sharper increase in the prevalence of direct CSW's as compared to that of indirect CSW's (see MOPH HIV Sentinel Surveillance, 1993–2000). The group of unpartnered young men is also divided into a class of highly sexually active unpartnered young men who are more likely to visit brothels and a class of low sexually active young men who are more likely to go to indirect CSW's (and with less frequency). As unpartnered young men become married, their spouses and offspring become exposed to the danger of HIV infections. With this much more complicated model, we wish to study the change of behaviour of the CSW's as well as the sexually active men who are crucial in the spread of the epidemic within the general population.

We give the model in detail in Section 2. The disease-free model is discussed in Section 3 with complete analysis. Since the general model is too complicated for analysis, Sections 4 and 5 will be devoted to discussing the respective local stability results of the disease-free equilibrium (DFE) of the model in two special cases. Namely, one with preferred mixing and no change of behaviour by either CSW's or their customers, and the other with change of behaviour but restricted mixing. In each case, we derive the basic reproduction number [see, e.g., Jacques *et al*. (1988) or Hyman and Li (2000)] for the spread of disease. In the latter case, we will also use illustrative examples to demonstrate the possible effect of behaviour change on the epidemic. Some general remarks are given in Section 6.

## 2. The Model

We begin by listing the key assumptions of the model:

### I. *Structure*

(1) No homosexual or drug activity.
(2) Recruitment of CSW's proportional to the total number of sexual contacts at the time—a 'Supply and Demand' assumption.
(3) Two sexual activity classes for CSW's and unpartnered men.
(4) The highly sexually active men go to the brothels more often for contacts with the direct CSW's while the less active men have more contacts with the indirect CSW's [i.e., preferred mixing in Jacques *et al*. (1988)].
(5) Partnered persons and noncommercial females are not considered in this model.

### II. *Females*

(1) Direct CSW's have higher contact rate than the indirect CSW's, but lower transmission probability due to more frequent use of condoms (*Thailand Social Monitor*, 2000). The MOPH survey (2000) also shows lower condom use among low-income factory workers, who are more likely to frequent brothels, than military conscripts who are representative of the general young male population since they are drafted randomly every year from the male population of 21 year old males.
(2) Movement of direct to indirect CSW class depends on the change in the fraction of sexual activity of men seeking indirect CSW's.
(3) CSW's retire to exit from the sexual network described in the model.

**III.** ***Males***

(1) Constant recruitment rate for males.
(2) Movement from more sexually active class to less active class depends on prevalence of HIV infections in females.
(3) Migration out of the unpartnered class through pairing.

Discussion on the motivation for the assumptions regarding the basic structure of the model (not involving distinct activity classes) can be found in Busenberg *et al.* (1995).

Next the **model variables** are given below:

$F_l$—Susceptible women of sexual activity group $l$ where

$l = 1$: (low activity) indirect CSW's
$l = 2$: (high activity) direct CSW's.

$f_l$—Infected women of sexual activity group $l$ where $l = 1, 2$ are the same as above.

$M_l$—Susceptible men of sexual activity group $l$ where

$l = 1$: (low activity) single men
$l = 2$: (high activity) single men.

$m_l$—Infected men of sexual activity group $l$ where $l = 1, 2$ are the same as above.

The **model parameters** are listed below:

$c_{jl}$: Contact rates of susceptibles of sex $j$, activity group $l$, with $c_{j1} < c_{j2}$.
$\bar{c}_{jl}$: Contact rates of infectives of sex $j$, activity group $l$, with $\bar{c}_{j1} < \bar{c}_{j2}$.
$\beta_{jlk}$: Transmission probability per contact of individual of sex $j$ and activity group $l$ to opposite sex of activity group $k$, with $\beta_{jlk} > \beta_{jmn}$ if $l + k < m + n$. i.e., the transmission probability is lower if at least one of individuals involved is of higher activity groups.
$\alpha$: Proportionality constant of the number of newly recruited CSW's to the total number of sexual contacts required by males at time $t$, $N_m(t)$, with $\alpha \in (0, 1)$.
$\theta$: Fraction of indirect CSW's needed in order to maintain equilibrium in the fraction of men seeking indirect CSW's.
$\rho_{fl}, \bar{\rho}_{fl}$: Retirement rate of susceptible and infected CSW's, respectively, of activity group $l$.
$\sigma_l, \bar{\sigma}_l, l = 1, 2$: Pairing rate of susceptible and infected single men, respectively, of group $l$.
$\mu_l, \bar{\mu}_l$: Removal (due to death, AIDS, etc.) rate of susceptibles and infecteds of sex $l$.
$\gamma_f, \bar{\gamma}_f$: Rates of movement from direct to indirect CSW class (or from indirect to direct depending on the signs of $G_1$ and $G_2$) for susceptibles and infecteds, respectively.

$\gamma_m$, $\bar{\gamma}_m$: Rates of changed behaviour of susceptible and infected single men, respectively, from high to low sexual activity due to the epidemic. Hence the rates are proportional to the endemic fraction of CSW population. Alternatively, we could also consider a movement from low to high activity class in the event of eradication of disease, or even perhaps due to a false sense of security brought on by the success of the prevention measures. However, there is no evidence of this occurring in Thailand at the present (Hsieh, 2002).

$\delta$: Yearly constant recruitment rate of sexually active single men, $\delta \gg 1$.

$\phi$: Initial fraction of susceptible men who are of low activity group.

$p_{kl}$: The preference probability that a male of group $k$ will have contact with a CSW of group $l$, with $p_{k1} + p_{k2} \leq 1$ and $p_{22} > p_{21}$.

The preference probability, $p_{kl}$, requires some explanation. The basic idea is similar to 'preferred mixing' (Jacques *et al.*, 1988), except $p_{k1} + p_{k2}$ might be less than one due to the possibility of males having sexual contacts with noncommercial female (steady or casual) partners. If $p_{kl} = \delta_{kl}$, the Kronecker delta, all contacts of males of group $k$ are restricted to those of CSW's in group $k$ and we have 'restricted mixing'. That is, a male customer is either strictly a brothel-visitor or a customer of nonbrothel sex establishments only.

The incidence rates of new infections, $\lambda_{fk}$ and $\lambda_{mk}$, where the subscript $f$ denotes females and $m$ denotes males, are given by:

$$\lambda_{fk}(t) = c_{fk} \sum_{l=1}^{2} \frac{p_{lk}\bar{c}_{ml}\beta_{mlk}m_l(t)}{N_m(t)}, \qquad k = 1, 2.$$

$$\lambda_{mk}(t) = c_{mk} \sum_{l=1}^{2} \frac{p_{kl}\bar{c}_{fl}\beta_{flk}f_l(t)}{N_f(t)}, \qquad k = 1, 2.$$

The total numbers of sexual contacts for males and females $N_m$ and $N_f$, where $f$ denotes females and $m$ denotes males, are:

$$N_m(t) = \sum_{l=1}^{2} [c_{ml}M_l(t) + \bar{c}_{ml}m_l(t)]$$

$$N_f(t) = \sum_{l=1}^{2} [c_{fl}F_l(t) + \bar{c}_{fl}f_l(t)].$$

The equations for the model depicted in Fig. 4 are, with '$\prime$' denoting derivative with respect to time:

$$F_1'(t) = \theta\alpha N_m(t) - [\mu_f + \rho_{f1} + \lambda_{f1}(t)]F_1(t) + \gamma_f G_1(t) \times \left[\frac{c_{m1}M_1(t) + \bar{c}_{m1}m_1(t)}{N_m(t)} - \theta\right]^2, \tag{1}$$

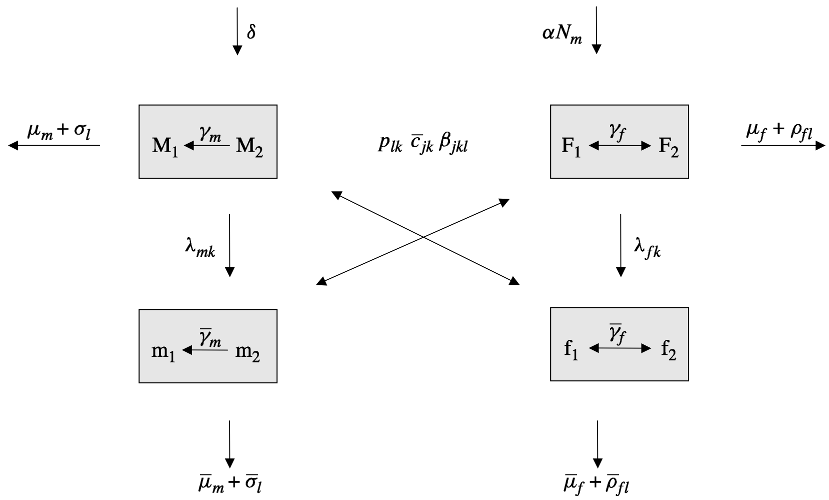



Figure 4. The model.

$$F_2'(t) = (1-\theta)\alpha N_m(t) - [\mu_f + \rho_{f2} + \lambda_{f2}(t)]F_2(t) - \gamma_f G_1(t) \times \left[\frac{c_{m1}M_1(t) + \bar{c}_{m1}m_1(t)}{N_m(t)} - \theta\right]^2, \quad (2)$$

$$M_1'(t) = \phi\delta - [\mu_m + \sigma_1 + \lambda_{m1}(t)]M_1(t) + \gamma_m M_2(t)\frac{\sum_{l=1}^2 f_l(t)}{\sum_{l=1}^2 F_l(t) + f_l(t)}, \quad (3)$$

$$M_2'(t) = (1-\phi)\delta - [\mu_m + \sigma_2 + \lambda_{m2}(t)]M_2(t) - \gamma_m M_2(t) \times \frac{\sum_{l=1}^2 f_l(t)}{\sum_{l=1}^2 F_l(t) + f_l(t)}, \quad (4)$$

$$f_1'(t) = -[\bar{\mu}_f + \bar{\rho}_{f1}]f_1(t) + \lambda_{f1}(t)F_1(t) + \bar{\gamma}_f G_1(t)\left[\frac{c_{m1}M_1(t) + \bar{c}_{m1}m_1(t)}{N_m(t)} - \theta\right]^2, \quad (5)$$

$$f_2'(t) = -[\bar{\mu}_f + \bar{\rho}_{f2}]f_2(t) + \lambda_{f2}(t)F_2(t) - \bar{\gamma}_f G_1(t)\left[\frac{c_{m1}M_1(t) + \bar{c}_{m1}m_1(t)}{N_m(t)} - \theta\right]^2, \quad (6)$$

$$m_1'(t) = \lambda_{m1}(t)M_1(t) - [\bar{\mu}_m + \bar{\sigma}_1]m_1(t) + \bar{\gamma}_m m_2(t)\frac{\sum_{l=1}^2 f_l(t)}{\sum_{l=1}^2 F_l(t) + f_l(t)}, \quad (7)$$

$$m_2'(t) = \lambda_{m2}(t)M_2(t) - [\bar{\mu}_m + \bar{\sigma}_2]m_2(t) - \bar{\gamma}_m m_2(t)\frac{\sum_{l=1}^2 f_l(t)}{\sum_{l=1}^2 F_l(t) + f_l(t)}, \quad (8)$$

where

$$G_1(t) = \begin{cases} F_2(t) & \text{if } \frac{c_{m1}M_1(t)+\bar{c}_{m1}m_1(t)}{N_m(t)} \geq \theta \\ -F_1(t) & \text{otherwise,} \end{cases}$$

$$G_2(t) = \begin{cases} f_2(t) & \text{if } \frac{c_{m1}M_1(t)+\bar{c}_{m1}m_1(t)}{N_m(t)} \geq \theta \\ -f_1(t) & \text{otherwise.} \end{cases}$$

The functions $G_i(t), i = 1, 2$ describe the change of behaviour by the susceptible and infected CSW's, respectively, as dictated by whether the fraction of men seeking indirect CSW's at time $t$ exceeds some equilibrium fraction value, $\theta$. More precisely, $G_1(t)$ equals $F_2(t)$ if the fraction of men seeking indirect CSW's at time $t$ exceeds $\theta$, indicating a flow from direct to indirect CSW's. Alternatively, it equals $-F_1$ if the fraction of men seeking indirect CSW's at time $t$ is below $\theta$, resulting in a flow from indirect to direct CSW's. Similarly for $G_2(t)$.

The squared terms in equations (1), (2), (5) and (6) represent the effect of 'overcrowding' (too much supply over demand) in the two CSW's groups which leads to the respective change of activity group from $F_1$ and $f_1$ to $F_2$ and $f_2$, or vice versa. Moreover, the squared terms also maintain the smoothness of the right-hand side of equations (1)–(8), which ensures that the system above is well-posed in the region $D_8 = \{F_i, M_i > 0, f_i, m_i \geq 0, i = 1, 2\}$ in the 8-dimensional space $\{(F_i, f_i, M_i, m_i), i = 1, 2\}$.

## 3. The Disease-Free Case

We start by considering the disease-free case where the population is free from HIV infections. Consequently we have $f_1 = f_2 = m_1 = m_2 = 0$, and the system (1)–(8) simplifies to

$$F_1'(t) = \theta\alpha N_m(t) - [\mu_f + \rho_{f1}]F_1(t) + \gamma_f G_1(t)\left[\frac{c_{m1}M_1(t)}{N_m(t)} - \theta\right]^2, \tag{9}$$

$$F_2'(t) = (1-\theta)\alpha N_m(t) - [\mu_f + \rho_{f2}]F_2(t) - \gamma_f G_1(t) \times \left[\frac{c_{m1}M_1(t)}{N_m(t)} - \theta\right]^2, \tag{10}$$

$$M_1'(t) = \phi\delta - [\mu_m + \sigma_1]M_1(t), \tag{11}$$

$$M_2'(t) = (1-\phi)\delta - [\mu_m + \sigma_2]M_2(t). \tag{12}$$

Note that $N_m$ now stands for $\sum_{l=1}^{2} c_{ml}M_l(t)$.

To find the equilibrium of the system, we will assume that the retirement rate of the two groups of CSW's are the same and the pairing rate of the two groups of young men are also the same. That is, $\rho_{f1} = \rho_{f2} = \rho$ and $\sigma_1 = \sigma_2 = \sigma$. We also redefine $\mu_m$ to be the former $\mu_m + \sigma$, $\mu_f$ to be the former $\mu_f + \rho$. Note that this simplification does not mean the retirement and pairing rates are neglected, just that they have the same significance as the removal by death or AIDS, etc. Consequently, $M_1'(t) = 0$ yields

$$\hat{M}_1 = \frac{\phi\delta}{\mu_m}, \tag{13}$$

and $M_2'(t) = 0$ yields

$$\hat{M}_2 = \frac{(1-\phi)\delta}{\mu_m}. \tag{14}$$

Moreover, by letting $\hat{N}_m = c_{m1}\hat{M}_1 + c_{m2}\hat{M}_2$, $(F_1(t) + F_2(t))' = 0$ implies

$$\hat{F}_1 + \hat{F}_2 = \frac{\alpha\hat{N}_m}{\mu_f}. \tag{15}$$

It follows that for $\frac{c_{m1}\hat{M}_1}{\hat{N}_m} \geq \theta$,

$$\hat{F}_1 = \frac{\alpha\hat{N}_m}{\mu_f} - \frac{(1-\theta)\alpha\hat{N}_m}{\gamma_f\left(\frac{c_{m1}\hat{M}_1}{\hat{N}_m} - \theta\right)^2 + \mu_f}, \tag{16}$$

$$\hat{F}_2 = \frac{(1-\theta)\alpha\hat{N}_m}{\gamma_f\left(\frac{c_{m1}\hat{M}_1}{\hat{N}_m} - \theta\right)^2 + \mu_f}. \tag{17}$$

If $\frac{c_{m1}\hat{M}_1}{\hat{N}_m} < \theta$,

$$\hat{F}_1 = \frac{\theta\alpha\hat{N}_m}{\gamma_f\left(\frac{c_{m1}\hat{M}_1}{\hat{N}_m} - \theta\right)^2 + \mu_f}, \tag{18}$$

$$\hat{F}_2 = \frac{\alpha\hat{N}_m}{\mu_f} - \frac{\theta\alpha\hat{N}_m}{\gamma_f\left(\frac{c_{m1}\hat{M}_1}{\hat{N}_m} - \theta\right)^2 + \mu_f}. \tag{19}$$

Therefore the equilibrium of the system exists and is unique. We now show that the unique equilibrium is globally asymptotically stable for all positive solutions. We drop the cap '^' in the equilibrium value $(\hat{F}_1, \hat{F}_2, \hat{M}_1, \hat{M}_2)$ and in $\hat{N}_m$ hereafter for sake of simplicity. The first result is the local stability of the unique equilibrium. The proof is in the Appendix.

**Lemma 1.** *The equilibrium point* $(F_1, F_2, M_1, M_2)$ *is locally asymptotically stable.*

We now give the global result, also proved in the Appendix.

**Theorem 2.** *The unique equilibrium* $(F_1, F_2, M_1, M_2)$ *is globally asymptotically stable for the system in the positive* 4*-dimensional region* $D_4^+ = \{(F_1(t), F_2(t), M_1(t), M_2(t)) \mid F_i(t), M_i(t) > 0, i = 1, 2\}$ *for a sufficiently small value of* $\gamma_f$.

We give a numerical example to illustrate the result in the disease-free state. Let $c_{f1} = 100$, $c_{f2} = 1000$, $c_{m1} = 10$, $c_{m2} = 50$, $\mu_f = 0.16$, $\mu_m = 0.09$, $\alpha = 0.0005$, $\delta = 500\,000$, $\theta = 0.5$, $\phi = 0.8$, $\gamma_f = 0.1$, $p_{11} = 0.5$, $p_{12} = 0.5$, $p_{21} = 0.1$, $p_{22} = 0.9$. The parameter values are consistent with the known literature for pre-1990 Thailand (before the HIV epidemic) and the assumptions of our model. For example, we let $\mu_f = 0.16 = 0.01 + 0.15$ where 0.01 is the mean mortality rate of a CSW approximated from Thai demographic data, and 0.15 is the retirement rate of the CSW. This value gives a mean working time of a CSW at approximately 4 years while a 1992 survey by Bhassorn *et al*. (1993) showed that the CSW's in Thailand worked an average of approximately 3.5 years (Busenberg *et al*., 1995).

We use the initial population values of $F_1 = 26\,000$, $F_2 = 18\,000$, $M_1 = 4000\,000$, $M_2 = 1000\,000$. The initial values for the CSW's are approximate values of government census numbers of 25 846 and 17 525, respectively, for 1971. The values for the male customers are deduced from the 1990 behaviour survey (Sittitrai *et al*., 1992) which reported that approximately 27% of the males of age 15–49 who responded said that they have had commercial sex in the last 12 months. Moreover, out of this group roughly one out of five (22%) are 'frequent' customers in the sense that they have had at least five different commercial sex partners during the last 12 months and used commercial sex often. We then use the demographic figure of approximately 18 million Thai men are of age 15–49 to arrive at our initial choice of population sizes. Figure 5 gives the numbers of CSW's obtained from numerical simulation of our theoretical model for 19 years, starting from 1971.

## 4. Full Model with Preferred Mixing and No Change in Activity Class

For the full model without change in behaviour, we have $\gamma_f = \bar{\gamma}_f = \gamma_m = \bar{\gamma}_m = 0$. Moreover we assume $\bar{\rho}_{f1} = \bar{\rho}_{f2} = \bar{\rho}$ and $\bar{\sigma}_1 = \bar{\sigma}_2 = \bar{\sigma}$ as we did for the susceptible groups in the previous section. We also redefine $\bar{\mu}_m$ to be the former $\bar{\mu}_m + \bar{\sigma}$, $\bar{\mu}_f$ to be the former $\bar{\mu}_f + \bar{\rho}$, and the system (1)–(8) simplifies to

$$F_1'(t) = \theta\alpha N_m(t) - [\mu_f + \lambda_{f1}(t)]F_1(t), \tag{20}$$

$$F_2'(t) = (1-\theta)\alpha N_m(t) - [\mu_f + \lambda_{f2}(t)]F_2(t), \tag{21}$$

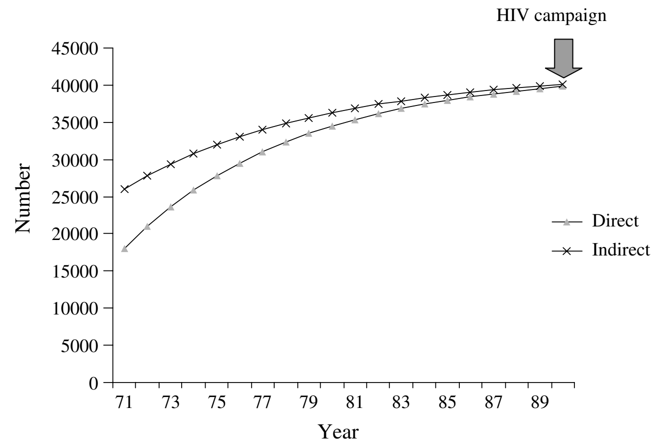



Figure 5. Simulation of the number of female sex workers in disease-free population in Thailand, 1971–1990.

$$M_1'(t) = \phi\delta - [\mu_m + \lambda_{m1}(t)]M_1(t), \tag{22}$$

$$M_2'(t) = (1-\phi)\delta - [\mu_m + \lambda_{m2}(t)]M_2(t), \tag{23}$$

$$f_1'(t) = -\bar{\mu}_f f_1(t) + \lambda_{f1}(t)F_1(t), \tag{24}$$

$$f_2'(t) = -\bar{\mu}_f f_2(t) + \lambda_{f2}(t)F_2(t), \tag{25}$$

$$m_1'(t) = \lambda_{m1}(t)M_1(t) - \bar{\mu}_m m_1(t), \tag{26}$$

$$m_2'(t) = \lambda_{m2}(t)M_2(t) - \bar{\mu}_m m_2(t). \tag{27}$$

Again the system is well-posed in $D_8$. The basic reproduction number $R_0$ is the expected number of secondary infections caused by an infected individual entering a population of all susceptibles [see, e.g., Anderson and May (1992)]. We have the following result for the basic reproduction number of the system. Let

$$R_{fkl}^2 = \frac{p_{kl}\bar{c}_{fl}\beta_{flk}}{\bar{\mu}_f}\frac{c_{mk}M_k}{N_m}, \qquad k, l = 1, 2;$$

$$R_{mkl}^2 = \frac{p_{kl}\bar{c}_{ml}\beta_{mkl}}{\bar{\mu}_m}\frac{c_{fk}F_l}{N_f}, \qquad k, l = 1, 2.$$

The basic reproduction number $R_0 \geq 0$ of the system (20)–(27) is given by

$$R_0^2 = R_{m12}^2R_{f12}^2 + R_{m21}^2R_{f21}^2 + R_{m11}^2R_{f11}^2 + R_{m22}^2R_{f22}^2 + R_{f11}^2R_{m12}^2R_{f22}^2R_{m21}^2 + R_{f21}^2R_{m11}^2R_{f12}^2R_{m22}^2 - R_{f11}^2R_{m11}^2R_{f22}^2R_{m22}^2 - R_{f21}^2R_{m12}^2R_{f12}^2R_{m21}^2. \tag{28}$$

Consequently we have the following lemma on local stability of DFE, the detailed proof is given in the Appendix.

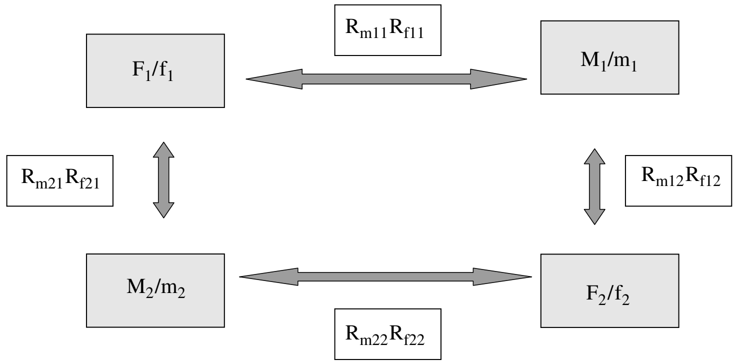



Figure 6. Infection cycles of length 2.

**Lemma 3.** *If $R_0 < 1$, the DFE $(F_1, F_2, M_1, M_2, 0, 0, 0, 0)$, with $F_1$, $F_2$, $M_1$, $M_2$ given in equations (13), (14) and (16)–(19), is locally asymptotically stable.*

To consider the epidemiological significance of the basic reproduction number, we define an *infection cycle* to be a path or a series of heterosexual transmissions of HIV culminating in a return to the starting point. Figures 6 and 7 give all possible infection cycles of length 2 and 4, while Fig. 8 shows all nonexistent cycles of length 4, i.e., all paths of infections involving the four groups that do *not* return to the starting point. Using this terminology, the square of the basic reproduction number $R_0^2$ is the sum of squares of the number of secondary infections produced by all possible infection cycles minus those produced by the nonexistent infection cycles. The result intuitively resembles the situation when one counts the number of elements in a union of sets, where we add all elements of the sets minus the elements in the intersections. A global stability result is difficult to obtain and will not be discussed here.

## 5. Full Model with Restricted Mixing and Change in Activity Class

Now we consider the full model with $p_{ij} = \delta_{ij}$, the Kronecker delta. That is, the males of group $k$ only have contact with CSW's of group $k$. This is called 'restricted mixing' by Jacques *et al.* (1988). Consequently we have $p_{12} = p_{21} = 0$ in the incidence rates $\lambda_{fk}$ and $\lambda_{mk}$ of the system (1)–(8). We first prove some preliminary result useful in obtaining the basic reproduction number for this case.

**Lemma 4.** (i) *For $\frac{c_{m1}M_1}{N_m} \geq \theta$.*

*If $R_{f11}^2 R_{m11}^2 < 1$ and $R_{f22}^2 R_{m22}^2 - \frac{\bar{\gamma}}{f}(c_{m1}M_1 - \theta N_m)^2 N_m^2 \bar{\mu}_f < 1$ where $R_{fkl}^2$ and $R_{mkl}^2$ are as defined in Section 4, then the $DFE(F_1, F_2, M_1, M_2, 0, 0, 0, 0)$ is locally asymptotically stable.*

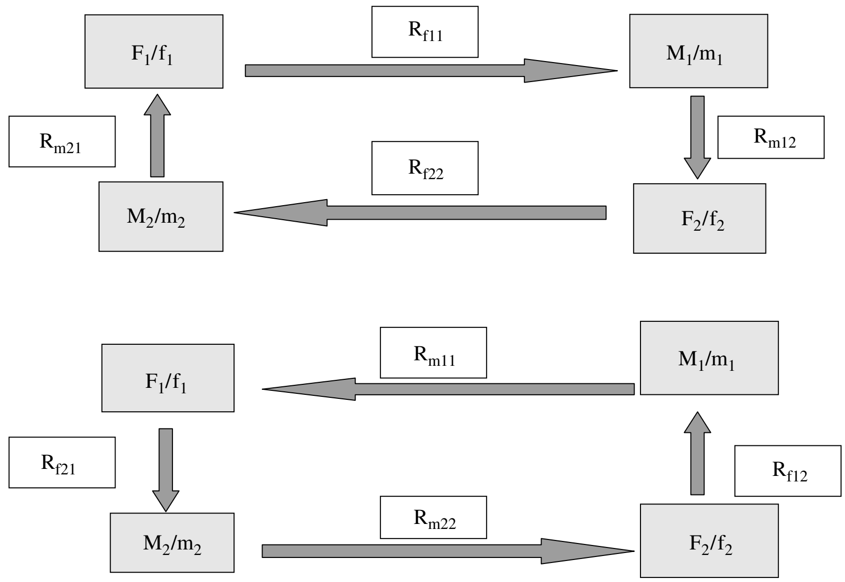



Figure 7. Infection cycles of length 4.

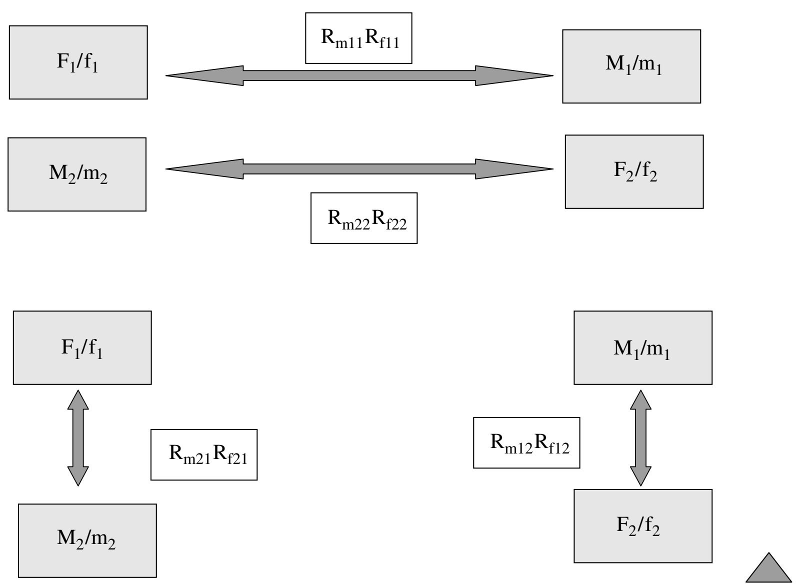



Figure 8. Nonexistent infection cycles of length 4.

(ii) *For* $\frac{c_{m1}M_1}{N_m} < \theta$.

*If* $R^2_{f22}R^2_{m22} < 1$ *and* $R^2_{f11}R^2_{m11} - \frac{\bar{\gamma}}{f}(c_{m1}M_1 - \theta N_m)^2 N_m^2 \bar{\mu}_f < 1$ *where* $R^2_{fkl}$ *and* $R^2_{mkl}$ *are as defined in Section 4, then the* $DFE(F_1, F_2, M_1, M_2, 0, 0, 0, 0)$ *is locally asymptotically stable.*

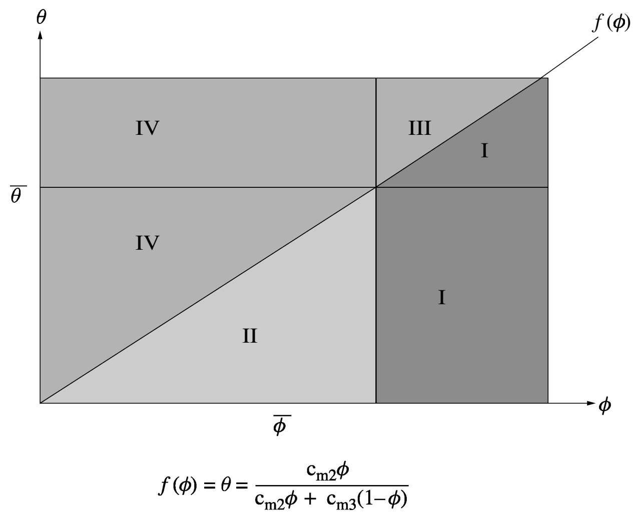



Figure 9. Basic reproduction number for restricted mixing: R1 for Region I, R2 for Region II, R3 for Region III, and R4 for Region IV.

The proof is also given in the Appendix.

Let us now define $R_{01}^2 = R_{f11}^2 R_{m11}^2$ and $R_{02}^2 = R_{f22}^2 R_{m22}^2$ and consider the case where $\frac{c_{m1}M_1}{N_m^2} = \theta$. It is easy to show that for $\phi \in [0, 1]$, $H(\phi) = R_{02}^2 - R_{01}^2$ is a strictly decreasing function of $\phi$ in [0, 1]. Moreover, $H(0)H(1) < 0$ hence there exists a unique number $\bar{\phi} \in (0, 1)$ at which $H(\bar{\phi}) = 0$. We also let $\bar{\theta} = \frac{c_{m1}M_1}{N_m^2} |_{\phi=\bar{\phi}}$ so that $\bar{\theta} \in (0, 1)$.

The expression for the basic reproduction number depends on $\bar{\phi}$ and $\bar{\theta}$. More precisely, if $\frac{c_{m1}M_1}{N_m} \geq \theta$,

$$R_0 = \begin{cases} R_1 & \text{if } 0 \leq \bar{\phi} \leq \phi \\ R_2 & \text{if } 0 \leq \phi < \bar{\phi}. \end{cases}$$

If $\frac{c_{m1}M_1}{N_m} < \theta$,

$$R_0 = \begin{cases} R_3 & \text{if } 0 \leq \bar{\phi} \leq \phi \\ R_4 & \text{if } 0 \leq \phi < \bar{\phi}, \end{cases}$$

where $M_1$ and $N_m$ are the disease-free numbers of $M_1(t)$ and $N_m(t)$, respectively, in Section 3 and

$$R_1^2 = R_{f11}^2 R_{m11}^2, \qquad R_4^2 = R_{f22}^2 R_{m22}^2,$$

$$R_2^2 = R_{f22}^2 R_{m22}^2 - \frac{\bar{\gamma}}{f}(c_{m1}M_1 - \theta N_m)^2 N_m^2 \bar{\mu}_f,$$

$$R_3^2 = R_{f11}^2 R_{m11}^2 - \frac{\bar{\gamma}}{f}(c_{m1}M_1 - \theta N_m)^2 N_m^2 \bar{\mu}_f.$$

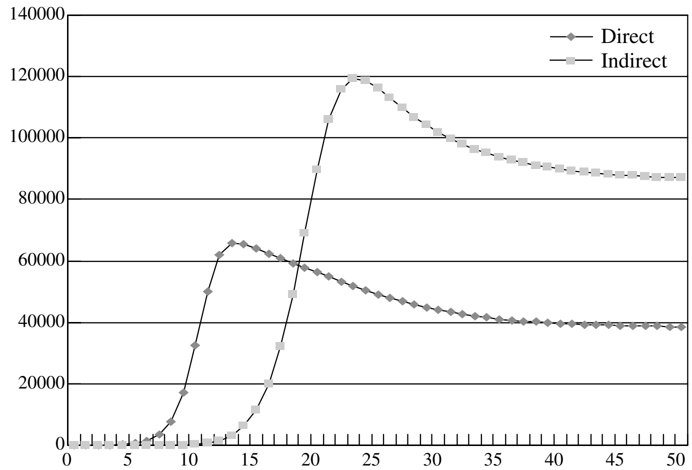



Figure 10. Simulation of model with behaviour change and restricted mixing, $R_0 = 5.9697$.

Let $\frac{c_{m1}M_1}{N_m} = f(\phi) = c_{m1}\phi/[c_{m1}\phi + c_{m2}(1-\phi)]$ from equations (13) and (14). It follows that the value of the basic reproduction number is as given in Fig. 9 in the rectangular region of $\phi, \theta \in [0, 1]$, with Region I denoting where $R_1$ is the basic reproduction number, Region II denoting $R_2$, Region III denoting $R_3$, and Region IV denoting $R_4$.

Note that for a fixed $\theta \in [0, 1]$ not equal to $\bar{\theta}$, the basic reproduction number changes twice as the value of $\phi$ increases from 0 to 1. For $\theta = \bar{\theta}$, there is only one switch. Similar switching occurs if we fix $\phi \in [0, 1]$ and let $\theta$ vary from 0 to 1.

To illustrate the results above, we again use numerical examples. First we let $p_{ij} = \delta_{ij}$, the Kronecker delta. We use the initial population values of $F_1 = 45\,000$, $F_2 = 41\,000$, $M_1 = 4000\,000$, $M_2 = 1000\,000$. The initial values for male customers are the same as in the previous section. The initial values for the CSW's are approximately the census numbers of Thai MOPH in 1990, the onset of HIV among Thai CSW's. All other parameter values given in Section 3 are used similarly. Moreover, we give the following set of epidemiological parameters for HIV in Thailand: $\bar{c}_{f1} = 100$, $\bar{c}_{f2} = 850$, $\bar{c}_{m1} = 10$, $\bar{c}_{m2} = 50$, $\bar{\mu}_f = 0.25$, $\bar{\mu}_m = 0.13$, $\beta_{f11} = 0.05$, $\beta_{f12} = \beta_{f21} = 0.02$, $\beta_{f22} = 0.005$, $\beta_{m11} = 0.08$, $\beta_{m12} = \beta_{m21} = 0.004$, $\beta_{m22} = 0.01$. The parameter values used are reasonable values taken from available literature. For example, in a study of 21 year old military conscripts whose principal mode of HIV transmission is sex with CSW's, Mastro *et al*. (1994) estimated the female-to-male HIV-1 transmission probability per sexual contact to be 0.056. Hence we let $\beta_{f11} = 0.05$ since condoms are used much less frequently by indirect CSW's. For the behaviour change, we let $\bar{\gamma}_f = 0.1$, $\gamma_m = \bar{\gamma}_m = 0.1$. Consequently, we have $\bar{\theta} = 0.6909$ and $\phi^* = 0.9179$. Thus we have $\theta = 0.5 < \bar{\theta}$ and $\phi = 0.8 < \bar{\phi}$. Moreover, $f(\phi) = 0.4444$ and

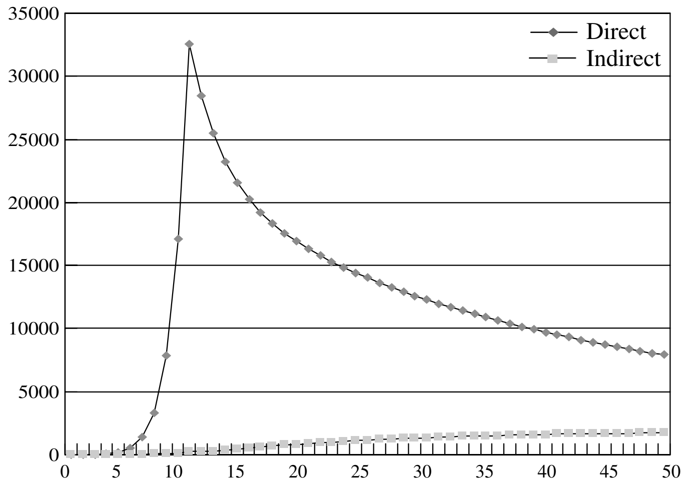



Figure 11. Simulation of model with behaviour change and restricted mixing, $R0 = 0.9626$.

consequently the example falls in Region IV and $R_0 = R_4 = 5.9697$. Hence the system approaches the endemic equilibrium in the numerical simulation given in Fig. 10.

Note that for this case, $\bar{\gamma}_f$ does not appear in the basic reproduction number for Region IV and hence has no bearing on the dynamics of the system. Again, if after 10 years of the epidemic we decrease $\beta_{m22}$ to 0.002 and $\beta_{f22}$ to 0.001, then $\bar{\theta} = 0.7143$ and $\bar{\phi} = 0.9259$. Given that $\phi$ and $\theta$ remain the same, we are again in Region IV and $R_0 = 0.9626$ which is less than unity. Consequently in the simulation shown in Fig. 11, the same initial populations would approach the DFE. Note that it is possible to choose the parameters so that if the source of lowly active males is not sufficiently high ($\phi < \phi^*$) and the changing of behaviour by males from the high to low activity group occurs rarely, the infected indirect CSW and lowly active male customer groups will be extinct. A similar statement holds for the high activity classes.

## 6. Concluding Remarks

From the results on basic reproduction number obtained and the numerical examples given in this work, we make the following epidemiologically significant conclusions:

(1) The basic reproduction number for the model without behaviour change given in equation (28), Section 4, adds the expected number of secondary infections for all possible paths of infection involving infection cycles of length 2 (Fig. 6) or 4 (Fig. 7), minus those of nonexisting infection cycles of length 4 (Fig. 8) which are intersections of cycles of length 2 and hence have already been accounted

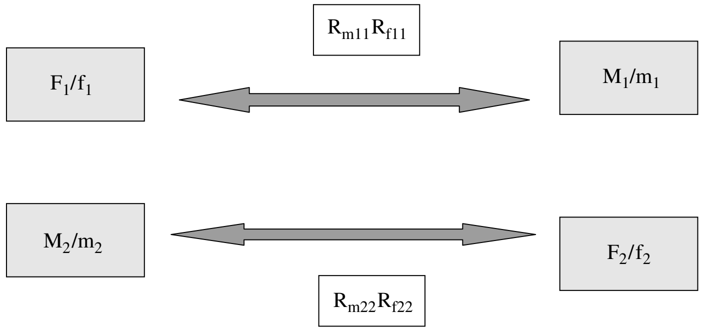



Figure 12. Infection cycles for restricted mixing.

for previously in the secondary infections caused by infection cycles of length 2. It accounts for all possible secondary infections without any redundant counting.

(2) The basic reproduction number obtained for the 'restricted mixing' model in Section 5 for the case of $p_{ij} = \delta_{ij}$ is determined by the values of $\theta$ and $\phi$ (Fig. 9). Note that $R_1$ gives the number of secondary infections caused by an individual (male or CSW) in the low activity class infection cycle without change of behaviour by a CSW, while $R_3$ is the number of secondary infections caused by an individual in the low activity class *with* CSW's switching activity group. $R_4$ and $R_2$ are the respective numbers for the high activity class infection cycle (Fig. 12). Note that although $\theta$ and $\phi$, which describe the relative sizes of the two classes of CSW's and male customers in the disease-free state, do not appear in the expressions for basic reproduction number, the fact that their values determine the form of the basic reproduction number underscores their importance in the dynamics of the system in question.

(3) For the restricted mixing model, the change of activity class by CSW's, namely $\bar{\gamma}_f$, is more important for intervention purposes as it could only decrease the basic reproduction number, while the change of behaviour of the male customers $\bar{\gamma}_m$ does not appear directly in the expression for the basic reproduction number and hence only affects the dynamics indirectly through its influence on the relative size of low and high activity male groups. In fact a sufficiently large $\bar{\gamma}_f$ could effectively drive the population toward the DFE. It shows that the social dynamics of the sex industry is not just a simple 'supply and demand' mechanism where the demand of the male customers dictates the supply or, in the context of preventing the spread of HIV, suppresses the supply of CSW's. Hence the change of behaviour of the CSW's or the changing structure of the sex industry must be monitored closely.

(4) Due to the complicated form of the full model, complete analysis cannot be readily carried out. However, simulations seem to indicate that the stability is global. That is, the DFE is globally asymptotically stable in the 8-dimensional region $D_8 = \{F_i, M_i > 0,\ f_i, m_i \geq 0, i = 1, 2\}$ when it is locally stable, and there

exists a unique endemic equilibrium which is globally asymptotically stable in $D_8$ when the DFE is unstable. Furthermore, simulations also show that for the full model with preferred mixing and change of activity class, similar existence and uniqueness results also prevail. However, the actual expression for the basic reproduction number is hard to derive and prove given the complicated $8 \times 8$ Jacobian matrix involved.

(5) The removal rates, $\bar{\mu}_f$ and $\bar{\mu}_m$, which appeared in the denominator of the basic reproduction numbers on p. 12 and thereafter, account for the removal due to death or onset of AIDS, as well as for the removal due to pairing of single men and retirement of CSW's. Hence an increase in pairing of single men or retirement of CSW's, which results in a loss of infected individuals in the sexually active population, would contribute to a decrease in the basic reproduction number.

(6) The numerical examples using the Thai HIV data seem to indicate that with the change we observe in the sex industry in Thailand, the level of epidemic is still high with $R_0$ exceeding 5 (see Fig. 10). Hence a large change of activity class $\bar{\gamma}_f$, though perhaps helpful in alleviating the magnitude of the epidemic, might not be sufficient to reverse the spread of disease by decreasing $R_0$ down to less than one. But if the transmission probabilities of the CSW's and customers, $\beta_{f22}$ and $\beta_{m22}$ in Fig. 11, are sufficiently lowered, one can still change the course of the epidemic favourably. Therefore an effective intervention programme targeted toward the CSW's and their customers is still very much in need, despite the encouraging signs in decreasing numbers of new infections in the recent years.

(7) The data used for estimating the parameters pertaining to behaviour change and the initial values are highly unreliable and therefore the numerical result serves only as an illustration of the analytical result, not necessarily a true description of what really happened. The usefulness of mathematical analysis is to understand qualitatively the behaviour of the system in question. The formulae for the basic reproduction number in various cases give us the exact condition under which a change in the parameter values would result in change in the dynamics of the system in a qualitative way. The result on the regions where distinct basic reproduction numbers prevail (p. 15) also shows exactly when a change in initial group sizes will alter the long-term behaviour of the system in question.

(8) Finally, the government policy for prevention and control has been to decrease the number of sexual contacts (hence infections) in the sex industry through a decrease in the direct sex industry, as well the transmission probability through use of condoms. Both have been successful to a degree. Our analysis shows that, however, the latter is more important than the former and therefore should be targeted more strongly.

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## APPENDIX

***Proof of Lemma 1.*** Let $J_1$ be the Jacobian matrix of the system at $(F_1, F_2, M_1, M_2)$. It suffices to show all eigenvalues of $J_1$ have negative real parts. For $\frac{c_{m1}M_1}{N_m} \geq \theta$,

$$\mathbf{J_1} = \begin{pmatrix} -\mu_f & \gamma_f\left[\frac{c_{m_1}M_1}{N_m} - \theta\right]^2 & \begin{matrix} c_{m_1}\theta\alpha \\ +WM_2 \end{matrix} & \begin{matrix} c_{m_2}\theta\alpha \\ -WM_1 \end{matrix} \\ 0 & \begin{matrix} -\mu_f \\ -\gamma_f\left[\frac{c_{m_1}M_1}{N_m} - \theta\right]^2 \end{matrix} & \begin{matrix} c_{m_1}(1-\theta)\alpha \\ -WM_2 \end{matrix} & \begin{matrix} c_{m_2}(1-\theta)\alpha \\ +WM_1 \end{matrix} \\ 0 & 0 & -\mu_m & 0 \\ 0 & 0 & 0 & -\mu_m \end{pmatrix}$$

where

$$N_m = c_{m_1}M_1 + c_{m_2}M_2,$$

$$W = 2\gamma_f F_2 \frac{c_{m_1}c_{m_2}}{N_m^2}\left[\frac{c_{m_1}M_1}{N_m} - \theta\right].$$

$J_1$ is an upper triangular matrix, it follows that the eigenvalues of $J_1$ are $-\mu_f$, $-\mu_f - \gamma_f[\frac{c_{m_1}M_1}{N_m} - \theta]^2$, $-\mu_m$, and $-\mu_m$.

Since $\mu_f$, $\mu_m$, and $\gamma_f$ are all positive, we have

$$-\mu_f - \gamma_f\left(\frac{c_{m_1}M_1}{N_m} - \theta\right)^2 < 0$$

and hence all eigenvalues are negative. The proof is similar for the case $\frac{c_{m1}M_1}{N_m} < \theta$.

***Proof of Theorem 2.*** To show global stability, we make use of a result by Li and Muldowney [Proof of Theorem 2.1, Li and Muldowney (1995)] which states that it suffices to show that the system is competitive in the region $D_4^+$.

First we consider the case of $\gamma_f = 0$, subsequently $W = 0$ and we have a constant Jacobian

$$\mathbf{J_1} = \begin{pmatrix} -\mu_f & 0 & c_{m_1}\theta\alpha & c_{m_2}\theta\alpha \\ 0 & -\mu_f & c_{m_1}(1-\theta)\alpha & c_{m_2}(1-\theta)\alpha \\ 0 & 0 & -\mu_m & 0 \\ 0 & 0 & 0 & -\mu_m \end{pmatrix}.$$

If we let $H = \text{diag}(1, 1, -1, -1)$, $H\mathbf{J_1}H$ has nonpositive off-diagonal elements and the system is competitive.

For the case $\gamma_f > 0$ but small, we note that all terms in $\mathbf{J_1}$ involving $\gamma_f$ are linear in $\gamma_f$, hence by a continuity argument the stability holds for sufficiently small and positive $\gamma_f$.

***Proof of Lemma 3.*** The Jacobian matrix $J_2^*$ at the DFE has the form

$$\mathbf{J}_2^* = \begin{pmatrix} \mathbf{J_1} & \cdot \\ 0 & \mathbf{J_2} \end{pmatrix}$$

where $\mathbf{J_1}$ is the Jacobian matrix of the disease-free case and

$$\mathbf{J_2} = \begin{pmatrix} -\bar{\mu}_f & 0 & b_1 & b_2 \\ 0 & -\bar{\mu}_f & b_3 & b_4 \\ a_1 & a_2 & -\bar{\mu}_m & 0 \\ a_3 & a_4 & 0 & -\bar{\mu}_m \end{pmatrix}$$

with

$$a_1 = \frac{c_{m1} p_{11} \bar{c}_{f1} \beta_{f11} M_1}{N_f}$$

$$a_2 = \frac{c_{m1} P_{12} \bar{c}_{f2} \beta_{f21} M_1}{N_f}$$

$$a_3 = \frac{c_{m2} p_{21} \bar{c}_{f1} \beta_{f12} M_2}{N_f}$$

$$a_4 = \frac{c_{m2} p_{22} \bar{c}_{f2} \beta_{f22} M_2}{N_f}$$

$$b_1 = \frac{c_{f1} p_{11} \bar{c}_{m1} \beta_{m11} F_1}{N_m}$$

$$b_2 = \frac{c_{f1} p_{21} \bar{c}_{m2} \beta_{m21} F_1}{N_m}$$

$$b_3 = \frac{c_{f2} p_{12} \bar{c}_{m1} \beta_{m12} F_2}{N_m}$$

$$b_4 = \frac{c_{f2} p_{22} \bar{c}_{m2} \beta_{m22} F_2}{N_m}.$$

We note that $\lambda(\mathbf{J_2^*}) = \lambda(\mathbf{J_1}) \cup \lambda(\mathbf{J_2})$.

We also know that all eigenvalues of $\mathbf{J_1}$ have negative real parts and the eigenvalues of $J_2$ are $-\frac{1}{2}(\bar{\mu}_m + \bar{\mu}_f) \pm \frac{1}{2}\sqrt{(\bar{\mu}_m - \bar{\mu}_f)^2 + 2h + 2\sqrt{k}}$ and $-\frac{1}{2}(\bar{\mu}_m + \bar{\mu}_f) \pm \frac{1}{2}\sqrt{(\bar{\mu}_m - \bar{\mu}_f)^2 + 2h - 2\sqrt{k}}$, where

$$h = a_1 b_1 + a_4 b_4 + a_2 b_3 + a_3 b_2$$

$$k = (a_1b_1 - a_4b_4)^2 + (a_3b_2 - a_2b_3)^2 + 2(a_3b_1 + a_4b_3)(a_1b_2 + a_2b_4)$$
$$+ 2(a_3b_4 + a_1b_3)(a_2b_1 + a_4b_2).$$

Clearly $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4, \bar{\mu}_m$ and $\bar{\mu}_f$ are all positive. Let

$$s_1 = -\tfrac{1}{2}(\bar{\mu}_m + \bar{\mu}_f) + \tfrac{1}{2}\sqrt{(\bar{\mu}_m - \bar{\mu}_f)^2 + 2h + 2\sqrt{k}}$$
$$t_1 = -\tfrac{1}{2}(\bar{\mu}_m + \bar{\mu}_f) + \tfrac{1}{2}\sqrt{(\bar{\mu}_m - \bar{\mu}_f)^2 + 2h - 2\sqrt{k}}.$$

To show that all eigenvalues of $J_3^*$ have negative real parts, it suffices to show $s_1$ and $t_1$ have negative real parts. We will proceed by considering two separate cases.
Case 1. Assume $(\bar{\mu}_m - \bar{\mu}_f)^2 + 2h - 2\sqrt{k} < 0$, i.e., $\mathrm{Re}(t_1) < 0$.

The condition in Lemma 3 is equivalent to

$$\frac{k}{(2\bar{\mu}_m\bar{\mu}_f - h)^2} < 1.$$

Given that $k > 0$, we can take the square root of both sides of the inequality and obtain

$$2\sqrt{k} + 2h + (\bar{\mu}_m - \bar{\mu}_f)^2 < (\bar{\mu}_m + \bar{\mu}_f)^2.$$

Since $h > 0$, it is equivalent to

$$s_1 = -\tfrac{1}{2}(\bar{\mu}_m + \bar{\mu}_f) + \tfrac{1}{2}\sqrt{(\bar{\mu}_m - \bar{\mu}_f)^2 + 2h + 2\sqrt{k}} < 0.$$

Case 2. Assume $(\bar{\mu}_m - \bar{\mu}_f)^2 + 2h - 2\sqrt{k} > 0$, i.e., $s_1 > t_1$.

Again we make use of the equivalent condition

$$\frac{k}{(2\bar{\mu}_m\bar{\mu}_f - h)^2} < 1,$$

which combined with $(\bar{\mu}_m - \bar{\mu}_f)^2 + 2h - 2\sqrt{k} > 0$ implies $s_1 < 0$. Since $t_1 < s_1$, it follows that $t_1 < 0$ as well.

***Proof of Lemma 4.*** (i) For $\frac{c_{m1}M_1}{N_m} \geq \theta$, the Jacobian matrix $J_3^*$ at DFE has the form

$$\mathbf{J_3}^* = \begin{pmatrix} \mathbf{J_1} & \cdot \\ 0 & \mathbf{J_3} \end{pmatrix}$$

where $J_1$ is the Jacobian matrix of the disease-free case in Section 3.

It suffices to show that all eigenvalues of $J_3$ have negative real parts where

$$\mathbf{J_3} = \begin{pmatrix} -\bar{\mu}_f & c & b_1 & 0 \\ 0 & -\bar{\mu}_f - c & 0 & b_4 \\ a_1 & 0 & -\bar{\mu}_m & 0 \\ 0 & a_4 & 0 & -\bar{\mu}_m \end{pmatrix}$$

with

$$a_1 = \frac{c_{m1}\bar{c}_{f1}\beta_{f11}M_1}{N_f},$$

$$a_4 = \frac{c_{m2}\bar{c}_{f2}\beta_{f22}M_2}{N_f},$$

$$b_1 = \frac{c_{f1}\bar{c}_{m1}\beta_{m11}F_1}{N_m},$$

$$b_4 = \frac{c_{f2}\bar{c}_{m2}\beta_{m22}F_2}{N_m},$$

$$c = \frac{\bar{\gamma}_f(c_{m1}M_1 - \theta N_m)^2}{N_m^2}.$$

Note that $\lambda(\mathbf{J_3^*}) = \lambda(\mathbf{J_1}) \cup \lambda(\mathbf{J_3})$.

We know that all eigenvalues of $\mathbf{J_1}$ have negative real parts. The eigenvalues of $J_3$ are $-\frac{1}{2}(\bar{\mu}_m + \bar{\mu}_f) \pm \frac{1}{2}\sqrt{(\bar{\mu}_m - \bar{\mu}_f)^2 + 4a_1b_1}$ and $-\frac{1}{2}(\bar{\mu}_m + \bar{\mu}_f + c) \pm \frac{1}{2}\sqrt{(\bar{\mu}_m - \bar{\mu}_f - c)^2 + 4a_4b_4}$. Since $a_1, a_4, b_1, b_4, c, \bar{\mu}_m$ and $\bar{\mu}_f$ are all positive, we only need to show that

$$-\frac{1}{2}(\bar{\mu}_m + \bar{\mu}_f) + \frac{1}{2}\sqrt{(\bar{\mu}_m - \bar{\mu}_f)^2 + 4a_1b_1} < 0$$

and

$$-\frac{1}{2}(\bar{\mu}_m + \bar{\mu}_f + c) + \frac{1}{2}\sqrt{(\bar{\mu}_m - \bar{\mu}_f - c)^2 + 4a_4b_4} < 0.$$

The first condition in Lemma 4 is

$$\bar{\mu}_m\bar{\mu}_f > a_1b_1$$

which is equivalent to

$$-\frac{1}{2}(\bar{\mu}_m + \bar{\mu}_f) + \frac{1}{2}\sqrt{(\bar{\mu}_m - \bar{\mu}_f)^2 + 4a_1b_1} < 0.$$

Similarly, the second condition in the lemma is

$$\bar{\mu}_m\bar{\mu}_f > a_4b_4 - \bar{\mu}_m c$$

which is equivalent to

$$-\frac{1}{2}(\bar{\mu}_m + \bar{\mu}_f + c) + \frac{1}{2}\sqrt{(\bar{\mu}_m - \bar{\mu}_f - c)^2 + 4a_4b_4} < 0.$$

So all eigenvalues of $J_2$, and subsequently $J_2^*$, have negative real parts.

(ii) Similarly for the case $\frac{c_{m1}M_1}{N_m} < \theta$.

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